University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 10

This problem set will be discussed on **Apr. 22**.

Problem 1: Syndrome Measurement for the Bit Flip Code

The three-qubit bit flip code protects a single qubit against bit flip (X) errors by encoding

$$|0\rangle \to |0_L\rangle \equiv |000\rangle, \qquad |1\rangle \to |1_L\rangle \equiv |111\rangle,$$

so that a general state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is encoded as $|\psi_L\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$. Each of the three qubits is then sent through an independent bit flip channel, which flips a qubit with probability p.

In this problem you work through the syndrome measurement procedure using the parity-check observables Z_1Z_2 and Z_2Z_3, and determine when the code improves the fidelity of storage.

a) The syndrome table.

Error detection in the bit flip code is performed by measuring the two observables $Z_1Z_2 \equiv Z\otimes Z\otimes I$ and $Z_2Z_3 \equiv I\otimes Z\otimes Z$. Each has eigenvalues ± 1.

i) Show that for any computational basis state $|xyz\rangle$,

$$Z_1Z_2\,|xyz\rangle = (-1)^{x\oplus y}\,|xyz\rangle, \qquad Z_2Z_3\,|xyz\rangle = (-1)^{y\oplus z}\,|xyz\rangle,$$

where $\oplus$ denotes addition modulo 2. Thus Z_1Z_2 "compares" qubits 1 and 2, yielding $+1$ if they agree and -1 if they differ.

ii) For each of the four cases (no error, X_1, X_2, X_3 acting on the encoded state), determine the eigenvalues of Z_1Z_2 and Z_2Z_3. Verify that the four syndrome outcomes are distinct.

Solution: **i)** Since $Z\,|0\rangle = |0\rangle$ and $Z\,|1\rangle = -\,|1\rangle$, we have $Z\,|x\rangle = (-1)^x\,|x\rangle$ for $x \in \{0,1\}$. Therefore

$$Z_1Z_2\,|xyz\rangle = (-1)^x(-1)^y\,|xyz\rangle = (-1)^{x+y}\,|xyz\rangle = (-1)^{x\oplus y}\,|xyz\rangle,$$

where the last step uses the fact that $(-1)^{x+y} = (-1)^{x\oplus y}$ for $x, y \in \{0,1\}$. Similarly, $Z_2Z_3\,|xyz\rangle = (-1)^{y\oplus z}\,|xyz\rangle$.

ii) The encoded state after a single-qubit error (or no error) is:

- No error: $\alpha\,|000\rangle + \beta\,|111\rangle$. Both terms have qubits 1 and 2 equal, and qubits 2 and 3 equal. Syndrome: $(Z_1Z_2, Z_2Z_3) = (+1, +1)$.
- X_1: $\alpha\,|100\rangle + \beta\,|011\rangle$. In both terms qubits 1 and 2 differ, while qubits 2 and 3 agree. Syndrome: $(-1, +1)$.
- X_2: $\alpha\,|010\rangle + \beta\,|101\rangle$. In both terms qubits 1 and 2 differ, and qubits 2 and 3 differ. Syndrome: $(-1, -1)$.
- X_3: $\alpha\,|001\rangle + \beta\,|110\rangle$. In both terms qubits 1 and 2 agree, while qubits 2 and 3 differ. Syndrome: $(+1, -1)$.

Error	Z_1Z_2	Z_2Z_3
None	$+1$	$+1$
X_1	-1	$+1$
X_2	-1	-1
X_3	$+1$	-1

All four outcomes are distinct, so the error can be uniquely identified among $\{I, X_1, X_2, X_3\}$.

b) Non-demolition property.

A key requirement for quantum error correction is that the syndrome measurement must not reveal any information about the encoded quantum state (i.e., about α and β).

i) Consider the error-free encoded state $|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$. The projector onto the $+1$ eigenspace of Z_1Z_2 is $P_+ = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I$. Show that $P_+|\psi_L\rangle = |\psi_L\rangle$, so that the outcome $+1$ occurs with certainty regardless of α and β, and the post-measurement state is unchanged.

ii) Now consider the state after a bit flip on qubit 1: $X_1|\psi_L\rangle = \alpha|100\rangle + \beta|011\rangle$. Show that measuring Z_1Z_2 yields -1 with certainty and that the state is unchanged. Explain why the measurement reveals information about the *error* but not about the *encoded state*.

Solution: i) We have $P_+ = (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I$. Acting on the two terms:

$$P_+(\alpha|000\rangle + \beta|111\rangle) = \alpha(|00\rangle\langle 00|\,|00\rangle)|0\rangle + \beta(|11\rangle\langle 11|\,|11\rangle)|1\rangle = \alpha|000\rangle + \beta|111\rangle = |\psi_L\rangle.$$

The probability of outcome $+1$ is $\|P_+|\psi_L\rangle\|^2 = |\alpha|^2 + |\beta|^2 = 1$, independent of α and β. The post-measurement state is $P_+|\psi_L\rangle / \|P_+|\psi_L\rangle\| = |\psi_L\rangle$, unchanged.

ii) The projector onto the -1 eigenspace is $P_- = (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$. Acting on the error state:

$$P_-(\alpha|100\rangle + \beta|011\rangle) = \alpha|100\rangle + \beta|011\rangle.$$

The probability of outcome -1 is 1, and the state is unchanged.

The syndrome measurement reveals that qubits 1 and 2 *differ* (indicating an error), but it does not distinguish between different values of α and β, because both $|100\rangle$ and $|011\rangle$ have the same parity for the first two qubits. The measurement extracts information about the *parity* of qubit pairs, which encodes the error pattern, not the encoded quantum information. This is what allows error correction without destroying the superposition.

c) Fidelity improvement.

Compare the minimum fidelity of the qubit with and without the bit flip code.

i) Without encoding, a qubit $|\psi\rangle$ sent through the bit flip channel has output $\rho' = (1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X$. Show that the minimum fidelity (over all $|\psi\rangle$) is $F_{\min} = \sqrt{1-p}$.

ii) With the three-qubit code followed by ideal syndrome measurement, correction, and decoding, show that the minimum fidelity is

$$F_{\min} = \sqrt{1 - 3p^2 + 2p^3}.$$

For what range of p does the code improve the minimum fidelity?

Hint[1].

Solution: i) The fidelity between the input state $|\psi\rangle$ and the output ρ' is

$$F(|\psi\rangle, \rho') = \sqrt{\langle\psi|\rho'\,|\psi\rangle} = \sqrt{(1-p) + p\,|\langle\psi|X\,|\psi\rangle|^2}.$$

The minimum occurs when $\langle\psi|X\,|\psi\rangle = 0$, which holds, for example, for $|\psi\rangle = |0\rangle$. Therefore

$$F_{\min} = \sqrt{1-p}.$$

ii) Error correction succeeds (recovering the exact state $|\psi\rangle$ after decoding) when at most one of the three qubits is flipped. This occurs with probability

$$p_{\text{succ}} = (1-p)^3 + 3p(1-p)^2 = 1 - 3p^2 + 2p^3.$$

The remaining events (two or three physical bit flips) occur with probability

$$p_{\text{fail}} = 3p^2(1-p) + p^3 = 3p^2 - 2p^3.$$

In each failure case, the syndrome-based recovery applies the wrong single-qubit correction (or no correction, for three flips), leaving the net operator $X_1X_2X_3$ on the code space. After decoding, this is just a logical bit flip X. Thus the decoded output state is

$$\rho_{\text{out}} = p_{\text{succ}}\,|\psi\rangle\langle\psi| + p_{\text{fail}} X\,|\psi\rangle\langle\psi|X.$$

Therefore

$$F(|\psi\rangle, \rho_{\text{out}}) = \sqrt{p_{\text{succ}} + p_{\text{fail}}\,|\langle\psi|X\,|\psi\rangle|^2}.$$

The minimum is attained when $\langle\psi|X\,|\psi\rangle = 0$, for example for $|\psi\rangle = |0\rangle$ or $|1\rangle$, so

$$F_{\min} = \sqrt{p_{\text{succ}}} = \sqrt{1 - 3p^2 + 2p^3}.$$

The code improves the minimum fidelity precisely when

$$1 - 3p^2 + 2p^3 > 1 - p.$$

For $0 \le p \le 1$, this is equivalent to

$$p - 3p^2 + 2p^3 > 0$$

or

$$p(1-p)(1-2p) > 0.$$

Since $0 \le p \le 1$, this gives $0 < p < 1/2$ for a *strict* improvement. The fidelities are equal at $p = 0$, $p = 1/2$, and $p = 1$.

Problem 2: The Shor Code

The three-qubit bit flip code corrects X errors but not Z errors. To correct phase errors (Z) as well, we need to also protect in the $|+\rangle/|-\rangle$ basis. The Shor code combines both ideas by

[1] Recall that $F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho\,|\psi\rangle}$. Minimize over $|\psi\rangle$ for part i). For part ii), note that successful error correction (zero or one bit flips) recovers the exact state, while two or three bit flips produce a logical X error on the code space after recovery, and hence after decoding.

concatenation into a nine-qubit code that corrects *any* single-qubit error — including errors that are not Pauli operators at all. This is possible because of a remarkable phenomenon called *error discretization*.

a) The phase flip code.

i) The bit flip channel is $\mathcal{E}_{\mathrm{BF}}(\rho) = (1-p)\rho + pX\rho X$, and the phase flip channel is $\mathcal{E}_{\mathrm{PF}}(\rho) = (1-p)\rho + pZ\rho Z$. Using $HXH = Z$, show that $\mathcal{E}_{\mathrm{PF}}(\rho) = H\,\mathcal{E}_{\mathrm{BF}}(H\rho H)\,H$ and explain what this implies: how can a bit flip code be turned into a phase flip code?

ii) The phase flip code encodes $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$, so a general state becomes $\alpha\,|+++\rangle + \beta\,|---\rangle$. Find two two-qubit observables (analogous to Z_1Z_2 and Z_2Z_3 for the bit flip code) that serve as syndrome measurements, and construct the syndrome table for no error and single-qubit errors Z_1, Z_2, Z_3.

Hint[2].

Solution: i)

$$\begin{aligned} H\,\mathcal{E}_{\mathrm{BF}}(H\rho H)\,H &= H\big[(1-p)(H\rho H) + pX(H\rho H)X\big]H \\ &= (1-p)\,\rho + p\,(HXH)\rho(HXH) = (1-p)\,\rho + p\,Z\rho Z = \mathcal{E}_{\mathrm{PF}}(\rho). \end{aligned}$$

The two channels are unitarily equivalent: the Hadamard gate exchanges $X \leftrightarrow Z$, so any bit flip code becomes a phase flip code by conjugating the encoding and decoding with $H^{\otimes n}$.

ii) Since $X\,|+\rangle = +\,|+\rangle$ and $X\,|-\rangle = -\,|-\rangle$, the operator X is diagonal in the $|\pm\rangle$ basis and "compares" qubits in that basis exactly as Z compares qubits in the $|0\rangle/\,|1\rangle$ basis. The syndrome observables are X_1X_2 and X_2X_3.

Using $Z\,|+\rangle = |-\rangle$ and $Z\,|-\rangle = |+\rangle$:

- No error: $\alpha\,|+++\rangle + \beta\,|---\rangle$. All qubits agree pairwise. Syndrome: $(+1,+1)$.
- Z_1: $\alpha\,|-++\rangle + \beta\,|+--\rangle$. Qubits 1 and 2 differ; qubits 2 and 3 agree. Syndrome: $(-1,+1)$.
- Z_2: $\alpha\,|+-+\rangle + \beta\,|-+-\rangle$. Qubits 1 and 2 differ; qubits 2 and 3 differ. Syndrome: $(-1,-1)$.
- Z_3: $\alpha\,|++-\rangle + \beta\,|--+\rangle$. Qubits 1 and 2 agree; qubits 2 and 3 differ. Syndrome: $(+1,-1)$.

Error	X_1X_2	X_2X_3
None	$+1$	$+1$
Z_1	-1	$+1$
Z_2	-1	-1
Z_3	$+1$	-1

The table has exactly the same structure as the bit flip syndrome table — a direct consequence of the channel equivalence.

b) The codewords.

The Shor code is constructed by *concatenation*: first encode using the phase flip code, then

[2] Recall that $X\,|+\rangle = |+\rangle$ and $X\,|-\rangle = -\,|-\rangle$, so X plays the role for the $|\pm\rangle$ basis that Z plays for the $|0\rangle/\,|1\rangle$ basis. Also use $Z\,|+\rangle = |-\rangle$ and $Z\,|-\rangle = |+\rangle$.

encode each qubit using the bit flip code.

i) The phase flip code maps $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$; each of these three qubits is then protected against bit flips by encoding $|\pm\rangle$ as

$$|+\rangle \to \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \qquad |-\rangle \to \frac{|000\rangle - |111\rangle}{\sqrt{2}}.$$

Write down the full codewords $|0_L\rangle$ and $|1_L\rangle$ for the Shor code.

ii) How many physical qubits are used? How many additional qubits initialized in $|0\rangle$ are required by the standard encoding circuit?

Solution: i) The phase flip encoding gives $|0\rangle \to |+++\rangle$ and $|1\rangle \to |---\rangle$. Replacing each $|\pm\rangle$ by its bit flip encoding:

$$|0_L\rangle = \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3} = \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}},$$
$$|1_L\rangle = \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3} = \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}.$$

ii) The code uses 9 physical qubits in total. Starting from one input qubit, the standard encoding circuit therefore uses 8 additional qubits initialized in $|0\rangle$.

c) Syndrome measurements in the Shor code.

The Shor code uses eight syndrome observables. For bit flips *within* each block of three qubits:

$$Z_1Z_2,\ Z_2Z_3,\ Z_4Z_5,\ Z_5Z_6,\ Z_7Z_8,\ Z_8Z_9.$$

For phase flips *between* blocks, use the two observables

$$X_{1\cdots 6} \equiv X_1X_2X_3X_4X_5X_6, \qquad X_{4\cdots 9} \equiv X_4X_5X_6X_7X_8X_9.$$

Work through the following two examples.

i) A bit flip on qubit 5 (i.e., X_5 acts on the encoded state). Which syndrome observables yield -1 and which yield $+1$? Identify the error and describe the recovery operation.

ii) A combined bit-and-phase flip X_1Z_1 acts on qubit 1. Determine all eight syndrome values. Show that the bit flip and phase flip syndromes detect the two components independently, and describe the recovery.

Hint[3].

Solution: i) X_5 flips qubit 5 (in block 2, containing qubits 4, 5, 6).

Bit flip syndromes: Within block 2, Z_4Z_5 compares qubits 4 and 5. After the flip, qubit 5 disagrees with qubit 4, so $Z_4Z_5 \to -1$. Similarly, $Z_5Z_6 \to -1$ since qubit 5 now disagrees with qubit 6. All other bit flip syndromes are $+1$, since blocks 1 and 3 are unaffected.

[3] Recall that operators commute or anti-commute with the syndrome observables depending on whether they share Pauli operators on the same qubits: $XZ = -ZX$ but $XX = ZZ = I$. An anti-commutation flips the eigenvalue of the syndrome observable.

Phase flip syndromes: X_5 commutes with both $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$, so both phase flip syndromes remain $+1$.

Z_1Z_2	Z_2Z_3	Z_4Z_5	Z_5Z_6	Z_7Z_8	Z_8Z_9	$X_{1\cdots6}$	$X_{4\cdots9}$
$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$

The pattern $Z_4Z_5 = Z_5Z_6 = -1$ (with all other bit flip syndromes $+1$) uniquely identifies a bit flip on qubit 5. Recovery: apply X_5.

ii) The error X_1Z_1 (equivalently Z_1X_1 up to a global phase) has both a bit flip and a phase flip component on qubit 1.

Bit flip syndromes: The X_1 component anti-commutes with Z_1Z_2, so $Z_1Z_2 \to -1$. It commutes with all other Z_iZ_j stabilizers. The Z_1 component commutes with every Z_iZ_j stabilizer, so it does not affect the bit-flip syndrome. Thus the bit-flip syndrome is exactly that of a bit flip on qubit 1: $Z_1Z_2 = -1$, $Z_2Z_3 = +1$, and all other Z-type syndromes are $+1$.

Phase flip syndromes: The Z_1 component anti-commutes with $X_1X_2X_3X_4X_5X_6$, so $X_{1\cdots6} \to -1$. It commutes with $X_4X_5X_6X_7X_8X_9$, so $X_{4\cdots9} = +1$. The X_1 component commutes with both X-type stabilizers, so it does not affect the phase-flip syndrome. Thus the phase-flip syndrome is that of a phase error in block 1.

Z_1Z_2	Z_2Z_3	Z_4Z_5	Z_5Z_6	Z_7Z_8	Z_8Z_9	$X_{1\cdots6}$	$X_{4\cdots9}$
-1	$+1$	$+1$	$+1$	$+1$	$+1$	-1	$+1$

The two syndrome types detect the X_1 and Z_1 components independently. Recovery: apply X_1 to correct the bit flip, and apply Z_1 to correct the phase flip. (Equivalently, one may apply Z_2 or Z_3, since on the block code space any single-qubit Z in block 1 has the same action.) For example, applying the recovery operator Z_1X_1 after the error X_1Z_1 gives

$$(Z_1X_1)(X_1Z_1) = I.$$

If one applies the two corrections in the opposite order, the net effect differs only by an overall minus sign, which is physically irrelevant.

d) Code limitations.

The Shor code corrects any *single-qubit* error. Here you explore what happens when two qubits in the same block are both flipped.

- **i)** Two bit flip errors occur in block 1: X_1X_2 acts on the encoded state. Determine all eight syndrome values. What does the error-correction procedure conclude, and what correction does it apply?
- **ii)** Apply the correction from part i) to the state after the X_1X_2 error. What is the net operation on block 1? Compute its effect on the block codewords $(|000\rangle + |111\rangle)/\sqrt{2}$ and $(|000\rangle - |111\rangle)/\sqrt{2}$, and explain what has happened to the encoded state.
- **iii)** If each qubit independently suffers a bit flip with probability p, what is the probability that a *given block* of three qubits has two or more flips, so that the inner bit-flip correction in that block fails? What is the probability that this happens in *at least one* of the three blocks? Is the single-block result consistent with your result from Problem 1(c)?

Hint[4].

[4] For ii), apply the net operation to $|000\rangle$ and $|111\rangle$ separately. Recall that $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$.

Solution: i) After X_1X_2, qubits 1 and 2 are both flipped. In block 1 (qubits 1, 2, 3):

- Z_1Z_2: both qubits are flipped, so they still agree $\rightarrow +1$.
- Z_2Z_3: qubit 2 is flipped, qubit 3 is not $\rightarrow -1$.

All other bit flip syndromes and both phase flip syndromes give $+1$ (since X_1X_2 commutes with the two X-type stabilizers).

Z_1Z_2	Z_2Z_3	Z_4Z_5	Z_5Z_6	Z_7Z_8	Z_8Z_9	$X_{1\cdots6}$	$X_{4\cdots9}$
$+1$	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$

The syndrome $(+1,-1)$ within block 1 is identical to the syndrome for a single X_3 error. The error-correction procedure misidentifies the error and applies X_3.

ii) The net operation on block 1 after the error X_1X_2 and the (incorrect) correction X_3 is $X_1X_2X_3$. We compute its action on the block-level codewords:

$$X_1X_2X_3\,|000\rangle = |111\rangle, \qquad X_1X_2X_3\,|111\rangle = |000\rangle.$$

Therefore:

$$X_1X_2X_3(|000\rangle + |111\rangle) = |111\rangle + |000\rangle = |000\rangle + |111\rangle,$$
$$X_1X_2X_3(|000\rangle - |111\rangle) = |111\rangle - |000\rangle = -(|000\rangle - |111\rangle).$$

So $X_1X_2X_3$ leaves the $|+\rangle$ block invariant but introduces a minus sign on the $|-\rangle$ block. In the Shor codewords, this means

$$X_1X_2X_3\,|0_L\rangle = |0_L\rangle, \qquad X_1X_2X_3\,|1_L\rangle = -\,|1_L\rangle.$$

The "corrected" state is therefore

$$\alpha\,|0_L\rangle - \beta\,|1_L\rangle,$$

so the encoded qubit has undergone a logical phase flip. Equivalently,

$$\alpha\,|0_L\rangle - \beta\,|1_L\rangle = Z_L(\alpha\,|0_L\rangle + \beta\,|1_L\rangle),$$

where Z_L denotes the logical phase-flip action on the code space. The encoded quantum information has been silently corrupted.

Note that $X_1X_2X_3$ commutes with all eight syndrome observables, so no stabilizer measurement detects it. This is a fundamental feature: $X_1X_2X_3$ is a *logical operator*, not a detectable error.

iii) The probability that a given block of three qubits has two or more bit flips is

$$q = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = 3p^2(1-p) + p^3 = 3p^2 - 2p^3.$$

The probability that this happens in at least one of the three blocks is therefore

$$1 - (1-q)^3 = 1 - \left(1 - 3p^2 + 2p^3\right)^3.$$

The single-block quantity q is exactly the uncorrectable error probability from Problem 1(c), as expected, because each block is itself the same three-qubit bit-flip repetition code.

Remark. Under *pure* physical X noise, the exact final logical-error probability of the full Shor code is not simply "at least one failed block", because one failed block acts like a logical Z_L, while two such block failures cancel. In that specific noise model, the logical error probability is the probability of an odd number of failed blocks:

$$3q(1-q)^2 + q^3.$$

Problem 3: Error Discretization

Physical noise rarely produces clean Pauli errors. A realistic channel might rotate a qubit by an arbitrary angle, or weakly entangle it with its environment. In this problem you show that syndrome measurement automatically converts any such continuous error into a discrete, correctable Pauli error.

a) Rotation error

Suppose qubit j of the encoded state $|\psi_L\rangle = \alpha\,|0_L\rangle + \beta\,|1_L\rangle$ undergoes a rotation error: the noise applies

$$R_x^{(j)}(\theta) = \cos\tfrac{\theta}{2}\, I - i \sin\tfrac{\theta}{2}\, X_j,$$

giving

$$R_x^{(j)}(\theta)\,|\psi_L\rangle = \cos\tfrac{\theta}{2}\ |\psi_L\rangle\ -\ i \sin\tfrac{\theta}{2}\ X_j\,|\psi_L\rangle.$$

i) The corrupted state is a superposition of two terms. What syndrome does each term have?

ii) When the syndrome is measured, what are the two possible outcomes, their probabilities, and the post-measurement state in each case?

iii) Apply the appropriate Pauli correction after each outcome. Show that $|\psi_L\rangle$ is recovered *exactly* in both cases, regardless of θ. What has happened to the continuous parameter θ?

Solution: i) The term $\cos(\theta/2)\ |\psi_L\rangle$ is proportional to the uncorrupted codeword, so it lies in the "no error" syndrome subspace: all eight syndrome observables give $+1$.

The term $-i\sin(\theta/2)\,X_j\,|\psi_L\rangle$ has exactly the same syndrome as a single physical bit flip on qubit j in the Shor code. More explicitly, within the three-qubit block containing qubit j:

- if j is the first qubit of the block, the two Z-type checks give $(-1,+1)$;
- if j is the middle qubit of the block, they give $(-1,-1)$;
- if j is the third qubit of the block, they give $(+1,-1)$.

All other Z-type checks give $+1$, and both X-type phase syndromes also give $+1$. Thus the two terms lie in orthogonal syndrome subspaces.

ii) Because the two terms lie in orthogonal syndrome subspaces, the syndrome measurement projects onto exactly one of them:

- "No error" syndrome, with probability $\cos^2\frac{\theta}{2}$. Post-measurement state: $|\psi_L\rangle$.
- "X_j error" syndrome, with probability $\sin^2\frac{\theta}{2}$. Post-measurement state: $X_j\,|\psi_L\rangle$.

(The overall factor $-i$ is only a global phase and has no physical effect.)

iii)

- "No error": apply no correction. State: $|\psi_L\rangle$.
- "X_j error": apply X_j. State: $X_j(X_j\,|\psi_L\rangle) = |\psi_L\rangle$.

In both cases the original encoded state is recovered exactly. The continuous angle θ survives only in the *probabilities* of the two syndrome outcomes. After syndrome measurement and correction, the quantum state itself has been discretized to one of two perfectly correctable Pauli-error branches.

b) Correcting Arbitrary Errors

So far, we have looked at discrete bit and phase flips. However, the Shor code in fact corrects *any* single-qubit error, continuous or non-unitary.

Consider a general noise channel acting on a single physical qubit k of the encoded state $\rho_L = |\psi_L\rangle\langle\psi_L|$:

$$\mathcal{E}(\rho_L) = \sum_j M_j \, \rho_L \, M_j^\dagger,$$

where each Kraus operator acts nontrivially only on qubit k, and $\sum_j M_j^\dagger M_j = I$. Here the physical qubit k is fixed throughout: the noise acts on one specified qubit, so the relevant Pauli error set is $\{I, X_k, Y_k, Z_k\}$ for that fixed k.

Prove that measuring the error syndrome and applying the corresponding Pauli correction perfectly recovers the original state $|\psi_L\rangle$. Hint[5].

Solution: Because $\{I, X, Y, Z\}$ is a basis for all 2×2 matrices, each Kraus operator acting on qubit k can be expanded as

$$M_j = \alpha_j I + \beta_j X_k + \gamma_j Y_k + \delta_j Z_k.$$

Thus the channel acts on the codespace through linear combinations of the four Pauli errors I, X_k, Y_k, and Z_k.

Let the eight stabilizer generators of the Shor code be denoted by S_m. For any single-qubit Pauli error $E \in \{I, X_k, Y_k, Z_k\}$, define its syndrome signs $s_m(E) \in \{+1, -1\}$ by

$$S_m E \, |\psi_L\rangle = s_m(E) \, E \, |\psi_L\rangle \qquad \text{for all code states } |\psi_L\rangle.$$

Because for a fixed qubit k the Shor code gives distinct syndromes for I, X_k, Y_k, and Z_k, we can define the projector onto the syndrome subspace of E by

$$P_E = \prod_{m=1}^{8} \frac{I + s_m(E) S_m}{2}.$$

Now let $F \in \{I, X_k, Y_k, Z_k\}$. Then

$$\frac{I + s_m(E) S_m}{2} \, F \, |\psi_L\rangle = \frac{1 + s_m(E) s_m(F)}{2} \, F \, |\psi_L\rangle.$$

If F has the same syndrome as E, each factor gives 1, so $P_E F \, |\psi_L\rangle = F \, |\psi_L\rangle$. If F has a different syndrome, then for at least one stabilizer generator S_m we have $s_m(F) = -s_m(E)$, and that factor gives zero. Therefore

$$P_E F \, |\psi_L\rangle = 0 \qquad \text{whenever } F \neq E.$$

So the four single-qubit Pauli error spaces are mutually orthogonal syndrome subspaces.

Applying P_E to $M_j \, |\psi_L\rangle$ therefore picks out only the E component:

$$P_E M_j \, |\psi_L\rangle = c_{j,E} \, E \, |\psi_L\rangle,$$

[5] Recall that $\{I, X, Y, Z\}$ forms a basis for all 2×2 matrices. Start by expanding each Kraus operator as $M_j = \alpha_j I + \beta_j X_k + \gamma_j Y_k + \delta_j Z_k$. Define the full syndrome projection operator P_E corresponding to a specific Pauli error $E \in \{I, X_k, Y_k, Z_k\}$, and show how it acts on the terms $M_j \, |\psi_L\rangle$.

where $c_{j,E}$ is one of $\alpha_j, \beta_j, \gamma_j, \delta_j$ depending on which Pauli error E is. Hence the unnormalized post-measurement state conditioned on syndrome E is

$$\begin{aligned}\widetilde{\rho}_E &= P_E\,\mathcal{E}(\rho_L)\,P_E \\ &= \sum_j P_E M_j \rho_L M_j^\dagger P_E \\ &= \sum_j |c_{j,E}|^2\, E\rho_L E^\dagger \\ &= p_E\, E\rho_L E^\dagger,\end{aligned}$$

where

$$p_E = \sum_j |c_{j,E}|^2$$

is the probability of obtaining syndrome E. After normalization, the post-measurement state is exactly

$$\rho_E = E\rho_L E^\dagger.$$

Finally, apply the Pauli correction $E^\dagger$. Since the chosen Pauli representatives $E \in \{I, X_k, Y_k, Z_k\}$ are unitary and Hermitian, $E^\dagger = E$ and $E^\dagger E = I$. Then

$$E^\dagger \rho_E E = E^\dagger E\,\rho_L\,E^\dagger E = \rho_L.$$

Thus syndrome measurement projects the arbitrary single-qubit error onto one of the discrete Pauli error sectors I, X_k, Y_k, or Z_k, and the corresponding correction restores the original encoded state exactly.